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Boundary-Layer Separation on Moving Walls Using an Integral Theory

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I. Introduction

THIS Note describes an integral method for obtaining separation from a moving wall. The technique originally was developed to obtain the force coefficients for a spinning cylinder in crossflow.[1] For handling the large boundary-layer reverse flows encountered on the spinning cylinder, the integral technique appears to have certain advantages over the finite-difference schemes.[2] With this method, the coupled integral-momentum and integral-energy equations are solved for the dependent variables, a shape factor K and the momentum thickness, in terms of the distance along the wall. The moving-wall similarity solutions provide a family of velocity profiles where all of the shape factors are known as functions of K and u_w/u_e (wall velocity to boundary-layer edge-velocity ratio); these shape factor relationships will be used to supply the values of the other shape factors appearing in the momentum and energy equations. These similarity solutions have been found for a wide range of u_w/u_e and pressure gradient parameter values.[3]

A new separation criterion is proposed here based on a singularity in the solution of the integral boundary-layer equations. The present criterion is a reasonable approximation to the Moore,[4] Rott,[5] and Sears[4] hypothesis for downstream moving walls and avoids some of the difficulties in using their criterion with the boundary-layer equations for upstream moving walls.[6]

Separation point predictions are compared with Vidal's[7] and Brady and Ludwig's[8] work. With the present approach, boundary layers with appreciable amounts of reverse flow can be calculated in a stable manner to separation.

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II. Integral-Momentum and Integral-Energy Equations

The Prandtl equations and boundary conditions for steady, constant-property, incompressible, two-dimensional, boundary-layer flow over moving walls are assumed.[2] The radius a of the cylinder will be taken as a fundamental length for nondimensionalizing the physical coordinates. Here, x is the nondimensional coordinate along the surface, y is the result of multiplying the normal physical coordinate by $Re_d^{\,1/2}/a$ where $Re_d = 2u_0\,a/\upsilon$. The kinematic viscosity is υ. All velocities are nondimensionalized with respect to u_0, the freestream velocity, and with the y component of the physical velocity being stretched by the factor $Re_d^{\,1/2}$.

Integrating partially with respect to y, the integral momentum equation may be written as

$$d\theta^2/dx = 2[2T - (2+H)\theta^2(du_e/dx)]/u_e \qquad (1)$$

The nondimensionalized edge velocity is given by u_e and θ is the momentum thickness nondimensionalized in the same way as the normal physical coordinate. Here H is the ratio of the displacement thickness to the momentum thickness and the skin friction factor T is given by the expression $(\theta/u_e)(\partial u/\partial y)_w$. The corresponding integral energy equation is

$$d(K^2\theta^2)/dx = 2[4K(L + u_w T/u_e) - 3K^2\theta^2(du_e/dx)]/u_e \quad (2)$$

Here K is the energy thickness divided by the momentum thickness and L is the dissipation integral nondimensionalized by multiplying with the factor $\tilde{\theta}/(\mu\tilde{u}_e^{\,2})$ where the superscript tilde denotes dimensioned quantities. The two differential equations, (1) and (2), are coupled and can be solved simultaneously for the dependent variables, K and θ^2, as a function of x with suitable initial conditions.

III. Flow Separation from Moving Walls

Moore,[4] Rott,[5] and Sears[4] have suggested that $\partial u/\partial y = 0$, $u = 0$ as a criterion (*MRS* criterion for separation on moving walls). The experimental observations of Swanson[9] and of Brady and Ludwig[8] for separation on the downstream moving part of a rotating cylinder tend to support the *MRS* criterion. Numerical calculations by Telionis and Werle[10] of the boundary layer on a tilted parabola with a downstream moving wall also show a tendency toward an *MRS* profile as separation is approached. They found that the boundary-layer thickness grows rapidly in this region suggestive of the approach to a singularity similar to that found in the fixed-wall case. However, the *MRS* criterion cannot be applied directly for walls moving upstream because the boundary-layer equations cannot be balanced.[6] Furthermore, inspection of the reversed-flow similarity profiles reveals no special characteristics which can be associated with *MRS*-like separation, nor is there any tendency toward these conditions.[3]

Tsahalis[12] recently has studied a nonsimilar upstream-moving wall problem using a finite-difference method. Since the boundary-layer equations generally are not valid in the immediate neighborhood of separation, he argues that the full Navier-Stokes equations may permit an *MRS* profile. Thus, he concludes that an approach to "near *MRS*-like" profiles may be a signal of incipient separation although further confirmation of this hypothesis is needed.

In the present investigation, it is assumed that the self-similar velocity profiles provide an adequate variation in

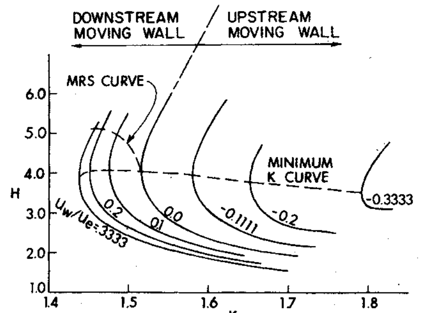

Fig. 1 *H* **vs** *K***-similarity solution values.**

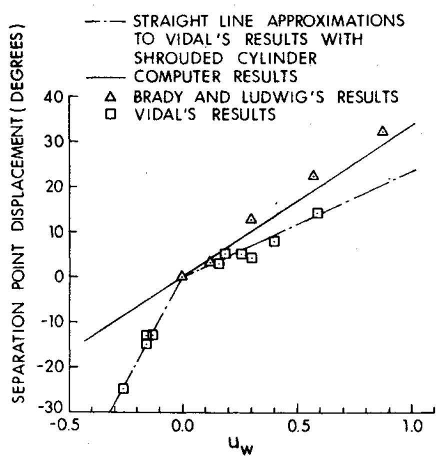

Fig. 2 Effect of wall velocity on apparent position of separation – comparison with experiment.

shape parameters including the approach to separation. Separation is determined under these conditions by the breakdown of the integral equations rather than some profile characteristic. As illustrated in Fig. 1, for certain intervals of K, H is double valued for both downstream and upstream moving walls. The lower branches of these curves are appropriate for use with the integral equations ahead of separation. In general, as the flow develops, K decreases and approaches a minimum value. A singularity is encountered in the solution of the integral equations when the minimum K is reached. Thus, the solutions with the minimum values of K will be designated as the separation solution for the integral technique. It may be observed that, for downstream moving walls, the *MRS* criterion corresponds to a point on the upper branch of the curves and cannot be reached with the integral method without passing through the singular point (except for the $u_w = 0$ case). The minimum K profiles resemble the *MRS* profiles but do not satisfy the *MRS* criterion. No *MRS* shape factors can be defined from the similarity solutions for $u_w/u_e < 0$ although the minimum K value and the singularity in the integral equations are found as direct extensions of the $u_w/u_e \geq 0$ situation.

The nature of the singularity in the integral equations can be developed by considering that $K = K(H, u_w/u_e)$; Eqs. (1) and (2) then can be transformed so that H is a dependent variable. Using the momentum equation, (1), to eliminate $d(\theta^2)/dx$, one obtains

$$dH/dx = E/[2\theta^2 u_e (\partial K/\partial H)_{u_w/u_e}] \quad (3)$$

where E is, in general, a finite nonzero quantity. However, from Fig. 1, $(\partial K/\partial H)_{u_w/u_e}$ approaches zero as the minimum values of K on the curves are approached. Thus, the points where K reaches a minimum on the H vs K curve correspond to singular points for the integral boundary-layer equations.

The order of the singularity can be obtained by expressing K as an analytical function of H near the singular value of K. One then may obtain that

$$H \simeq H_s - C_3 (x_s - x)^{1/2} \quad (4)$$

One sees that H (or equivalently, the displacement thickness) has a half-order singularity at x_s. Here, the singularity condition is denoted by the subscript s. Equation (4) may be utilized[1] also to obtain that the singularity found for the integral boundary-layer equations is similar to the Goldstein singularity.[11]

In summary then, the Goldstein-type singularity found with the integral boundary-layer equations has been chosen as the separation criterion for walls moving both upstream and downstream. A singularity also has been noted for moving walls using numerical techniques.[10,12] For downstream-moving walls, the profiles corresponding to separation for the integral method resemble the profiles that are near separation as obtained by finite-difference techniques.

IV. Separation Results Compared with Experiment

In Fig. 2, a comparison is made between the measurements of separation on a rotating cylinder by Vidal[7] and Brady and Ludwig.[8] The prescribed velocity distributions used with the integral method are given by wakeless potential flow corresponding to a nonrotating cylinder in crossflow and a sine-function approximation to experimental data obtained for the nonrotating cylinder in crossflow. The integral results are well approximated by a single linear function of the nondimensional wall velocity u_w.

Vidal's[7] measurements were obtained on a shrouded rotating cylinder to reduce unsteadiness near the separation point. Vidal[7] also obtained measurements on unshrouded cylinders; for downstream moving walls, the data obtained for both the shrouded and the unshrouded cylinder may be approximated by a single linear function. He used a smoke-flow visualization technique to estimate the position of the separation points which was defined as the point where the boundary layer thickened rapidly.

Brady and Ludwig[8] repeated Vidal's experiment with the same model and wind tunnel but using a hot-wire anemometer to assist in determining velocity profiles near separation; they felt that using only the displacement thickness criterion was inadequate to determine separation. Brady and Ludwig[13] later noted a weakness in their own treatment of the data for upstream moving walls. Thus, for upstream moving walls, the validity of both experiments is questionable.

Good agreement between the integral method and the data of Brady and Ludwig[8] is obtained for $u_w > 0$, whereas Vidal's[7] results do not agree as well at any velocity. However, both experiments show an almost linear relationship between separation point displacement and rotation rate as does the integral method. For upstream moving walls, Brady and Ludwig[8] found no separation displacement variation; this disagrees with both the integral analysis results and Vidal's[7] results.

The separation point displacement for downstream moving walls also has been investigated theoretically by Hartunian and Moore[14] utilizing Goldstein's stream-function formulation.[11] In this analysis, the shift in the separation point is shown to be linear with the wall velocity. It is noted that the separation point for the integral technique is a singular point as it is for Hartunian and Moore's[14] analysis.

V. Conclusions

Utilizing a two-equation integral technique, a boundary-layer separation criterion is developed for both upstream and downstream moving walls that corresponds to a singular point for the boundary equations. When the method is applied to the case of a rotating cylinder a linear displacement of the separation point with rotation rate is predicted. This result is in agreement with the limited experimental data available[7,8] and a theoretical analysis.[14]

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Orthogonality of Generally Normalized Eigenvectors and Eigenrows

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FREE vibration analysis of linear dynamic systems in the presence of viscous damping leads to the generalized eigenvalue problem

$$Q(\lambda)x \equiv [\lambda^2 A + \lambda B + C]x = 0 \tag{1}$$

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In this equation the matrices $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$ represent the inertia, damping and stiffness properties, respectively, of the system. They are of order $n \times n$ where n is the number of degrees of freedom considered in the analysis.

Equation (1) has, in general, $2n$ eigenvalues λ_r associated with $2n$ eigenvectors x_r and $2n$ eigenrows y_s^T.[1] They satisfy the equations

$$[\lambda_r^2 A + \lambda_r B + C]x_r = 0, \quad (r = 1,2,\ldots,2n) \tag{2}$$

and

$$y_s^T [\lambda_s^2 A + \lambda_s B + C] = 0, \quad (s = 1,2,\ldots,2n) \tag{3}$$

If the eigenvalues are all distinct, each eigenvector and eigenrow is uniquely defined to the extent of an arbitrary multiplier which is chosen to satisfy a convenient normalization criterion.

The eigenvectors and eigenrows are also orthogonal in the sense that, for $r \neq s$,

$$y_s^T [(\lambda_s + \lambda_r)A + B]x_r = 0, \ (r,s = 1,2,\ldots,2n) \tag{4}$$

For $r = s$, however, the left-hand side of Eq. (4) does not vanish, and it can be made equal to any desired value by adjusting the arbitrary multipliers of the normalized x_r and y_r^T. When normalization is done according to the criterion

$$y_r^T [2\lambda_r A + B]x_r = 1, \ (r = 1,2,\ldots,2n) \tag{5}$$

Eqs. (4) and (5) can be combined in the single matrix equation

$$\Lambda YAX + YAX\Lambda + YBX = I \tag{6}$$

where

$$X = \begin{bmatrix} x_1 & x_2 & \ldots & x_{2n} \end{bmatrix} \tag{7a}$$

$$Y = \begin{bmatrix} y_1^T \\ y_2^T \\ \vdots \\ y_{2n}^T \end{bmatrix} \tag{7b}$$

$$\Lambda = \operatorname{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_{2n}\} \tag{7c}$$

and

I is the unit matrix of order $2n$ (7d)

Under this particular normalization, it has been found[2] that each row of the eigenvectors modal matrix X is orthogonal to each column of the eigenrows modal matrix Y, that is to say

$$XY = 0 \tag{8}$$

Equation (8) is an interesting result which has found some useful applications in the theory of forced vibration of nonconservative systems.[3] Its validity relies, of course, on the adoption of Eq. (5) as a criterion for normalization. But in some cases, Eq. (5) is not very convenient, and other criteria are preferred. The question then arises as to whether an orthogonality relationship can still be found to replace Eq. (8) in such a case. This question is answered here by deriving the